Differential Inclusions on Wasserstein spaces

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Abstract

Optimal Control in Wasserstein spaces addresses control of systems with large number of agents. It is well known that for optimal control of ODEs, the differential inclisions theory provides useful tools to investigate existence of optimal controls, necessary optimality conditions and Hamilton-Jacobi-Bellman equations. Recently many models arising in social sciences use the framework of Wasserstein spaces, i.e. metric spaces of Borel probability measures endowed with the Wasserstein metric.

This talk is devoted to a recent extension given in [1] of the theory of differential inclusions to the setting of general Wasserstein spaces. Anchoring our analysis on novel estimates for solutions of continuity equations, a new existence result "à la Peano" for this class of dynamics, under mere Carathéodory regularity assumptions, is obtained. The latter is based on a set-valued generalisation of the semi-discrete Euler scheme proposed by Filippov to study ordinary differential equations with measurable right-hand sides. Substantial improvements to the earlier versions, see [3, 4], of the Filippov theorem, compactness and relaxation properties of the solution sets of continuity inclusions in the Cauchy-Lipschitz setting are given. This allows to get new results on existence of optimal controls and regularity of the value function.

In the second part of the talk necessary and sufficient conditions for the existence of solutions to state-constrained continuity inclusions in Wasserstein spaces, whose right-hand sides may be discontinuous in time, are provided, see [2]. These latter are based on a fine investigation of the infinitesimal behaviour of the underlying reachable sets, which heuristically amounts to showing that up to a negligible set, every admissible velocity can be realised as the metric derivative of a solution of the continuity inclusion, and vice versa. Building on these results, necessary and sufficient geometric conditions for the viability and invariance of stationary and time-dependent constraints, which involve a suitable notion of contingent cones in Wasserstein spaces, are established. Viability and invariance theorems in a more restrictive framework were already applied in [5], [6] to investigate stability of controlled continuity equations and uniqueness of solutions to HJB equations. The provided new tools allow to get similar results in general Wasserstein spaces.

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